Math 347H: Fundamental Math (H) HOMEWORK 8 **Due date: Nov 16 (Thu)**

1. (a) Prove that the set $P[\mathbb{Q}]$ of polynomials over $\mathbb{Q}$ is countable.

(b) Prove that the set of all algebraic reals is countable. You may use without proof that each polynomial has only finitely-many roots.

2. Prove in detail:

(a) $\mathscr{P}(\mathbb{N}) \cong 2^{\mathbb{N}}$.

(b) $\mathbb{N}^{\mathbb{N}} \hookrightarrow 2^{\mathbb{N}}$.

(c) $\mathbb{Q}^{\mathbb{N}} \cong \mathbb{N}^{\mathbb{N}}$.

3. Prove the following using any polynomial, rational, trigonometric, or other functions you like.

(a) $(0,1) \cong (1,+\infty)$.

(b) $(0,1) \cong (a,b)$ for any reals $a < b$.

(c) $(0,1) \cong \mathbb{R}$.

(d) $(0,1) \cong [0,1]$.

4. For sets $U \subseteq Y \times X$ and $y \in Y$, let $U_y := \{x \in X : (y,x) \in U\}$ and call it the *fiber of* U *over* y. When $Y = X$, we define the set $\text{AntiDiag}(U) := \{x \in X : x \notin U_x\}$ and call it the *antidiagonal of* U.

Cantor's Antidiagonaliation. *For any set* $U \subseteq X \times X$ *its antidiagonal is not one of its fibers, i.e.* $\text{AntiDiag}(U) \neq U_x$ *for every* $x \in X$.

5. For any (possibly uncountable) set X and any function $f : X \to \mathscr{P}(X)$, define a set $U \subseteq X \times X$ (think of it as a table indexed by X) such that $\text{AntiDiag}(U)$ is not in the image of f. Deduce Cantor's theorem that $X \not\twoheadrightarrow \mathscr{P}(X)$.

6. Perform the following base conversions of numbers and replace the question marks with your answers. Include your calculations.

(a) $536_7 = ?_{10}$.

(b) $1000000_2 = ?_8$.

(c) $20_{10} = ?_{16}$.

7. Describe the algorithm/procedure of writing a rational $\left[\frac{a}{b}\right]$ in its decimal representation. Prove that the latter is well-defined.